INFORMATION THEORY & CODING

Week 5 : Source Coding 1

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Review Summary

Theorem (AEP)

"Almost all events are almost equally surprising." Specifically, if $X_1, X_2, \ldots$ are i.i.d. $\sim p(x)$, then

$$-\frac{1}{n}\log p(X_1, X_2, \ldots, X_n) \to H(X) \text{in probability}.$$

Definition

The *typical set* $A_\epsilon^{(n)}$ is the set of sequences $x_1, x_2, \ldots, x_n$ satisfying

$$2^{-n(H(X)+\epsilon)} \leq p(x_1, x_2, \ldots, x_n) \leq 2^{-n(H(X)-\epsilon)}.$$

Review Summary

Properties of the typical set

1. If $(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}$, then $H(X) - \epsilon \leq -\frac{1}{n} \log p(x_1, x_2, \ldots, x_n) \leq H(X) + \epsilon$.
2. $\Pr[A_\epsilon^{(n)}] > 1 - \epsilon$ for n sufficiently large.
3. $|A_\epsilon^{(n)}| \leq 2^{n(H(X)+\epsilon)}$, where $|A|$ denotes the cardinality of the set A.
4. $|A_\epsilon^{(n)}| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$ for n sufficiently large.

Theorem

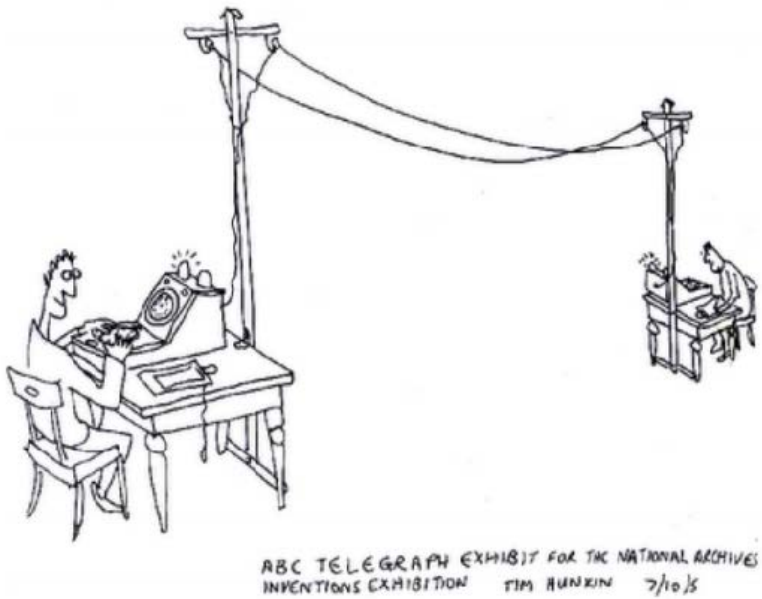
Let X^n be i.i.d. $\sim p(x)$. There exists a code that one-to-one maps sequences x^n of length n into binary strings with

$$E[\frac{1}{n}\ell(X^n)] \leq H(X) + \epsilon$$

for n sufficiently large.

Source Coding

Which horse won in the horse racing?



Source Coding

Which horse won in the horse racing?

X	Pr	Code I	Code II
0	1/2	000	0
1	1/4	001	10
2	1/8	010	110
3	1/16	011	1110
4	1/64	100	111100
5	1/64	101	111101
6	1/64	110	111110
7	1/64	111	111111

$$H(X) = -\sum p_i \log p_i = 2\text{bits}$$

Which code is better?

Source Coding (Data Compression)

- We interpret that $H(X)$ is the best achievable data compression.

- We want to develop practical lossless coding algorithms that approach, or achieve the entropy limit $H(X)$.

Terminology

X	Pr	Code I	Code II
0	$1/2$	000	0
1	$1/4$	001	10
2	$1/8$	010	110
3	$1/16$	011	1110
4	$1/64$	100	111100
5	$1/64$	101	111101
6	$1/64$	110	111110
7	$1/64$	111	111111

- Source alphabet $\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
- Code alphabet $\mathcal{D} = \{0, 1\}$.
- Codeword, e.g., 010 for $X = 2$ in Code 1.
- Codeword length, e.g., codeword length for Code 1 is 3.
- Codebook: all the codewords.

Source Coding

Notation (Alphabet Extension)

The set of all possible sequences based on a finite alphabet $\mathcal{D}$ is denoted by $\mathcal{D}^*$. E.g.,
$\mathcal{D} = \{0, 1\} \rightarrowtail \mathcal{D}^* = \{0, 1, 00, 01, 10, 11, 000, ...\}$.

Definition (Source Code)

Let $\mathcal{X}$ be the alphabet of a random variable X, and $\mathcal{D}$ be the alphabet of code. A *source code* C for the random variable X is a map

$$\begin{aligned} C: \quad & \mathcal{X} \to \mathcal{D}^* \\ & x \mapsto C(x) \end{aligned}$$

where $C(x)$ is the codeword associated with x. Let $\ell(x)$ denote the length of C(x).

Definition

The *expected length* $L(X)$ of a source code C for a random variable X with probability mass function $p(x)$ is

$$L(X) = E\ell(X) = \sum_{x \in \mathcal{X}} p(x)\ell(x).$$

X	Pr	Code I	Code II
0	1/2	000	0
1	1/4	001	10
2	1/8	010	110
3	1/16	011	1110
4	1/64	100	111100
5	1/64	101	111101
6	1/64	110	111110
7	1/64	111	111111

$$L_1(X) = 3$$
$$L_2(X) = 2$$

Source Coding Applications

- Magnetic recording: cassette, hard drive ...
- Speech compression
- Compact disk (CD)
- Image compression: JPEG

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Code efficiency $= H(X)/E[\ell(X)]$
- Which code is best? Would we prefer C_I or C_{II}?

  Consider C_I and decode string: 00001. It would come from $1, 2, 1, 2, 3$ or $2, 1, 2, 1, 3$ or $1, 1, 1, 1, 3$, or etc.

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Code efficiency $= H(X)/E[\ell(X)]$
- Which code is best? Would we prefer C_I or C_{II}?

  Consider C_{II} and decode string: 0011. It could be either $1, 1, 2, 2$ or $3, 4$.

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Consider C_{III}. Can we decode 1100000000?

  Yes. But if we only see a prefix, such as 11, we don't know until we see more bits to the end.

  $1100000000 = 3, 2, 2, 2, 2$

  $11000000000 = 4, 2, 2, 2, 2$

Source Coding: Set of codes

For $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{D} = \{0, 1\}$, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- Consider C_{IV}. This code seems at least feasible (since $E[\ell] \geq H$). Decoding seems easy: (e.g., $111110100 = 111, 110, 10, 0 = 4, 3, 2, 1$).

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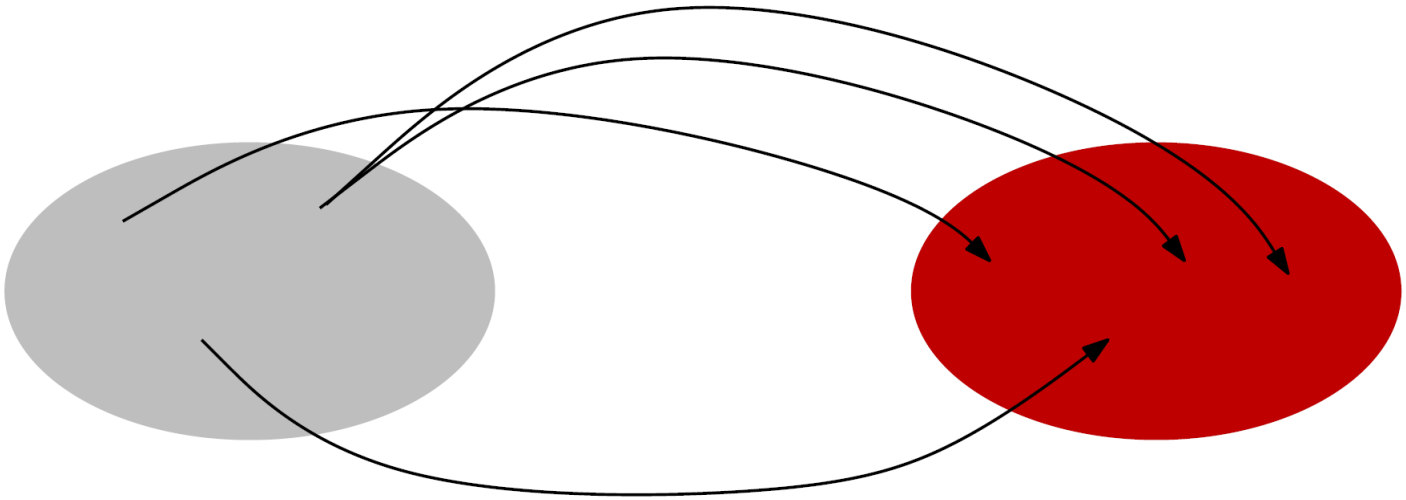
Source Coding: Code types

Definition (Nonsingular Code)

A code C is called *nonsingular* if every realization of $\mathcal{X}$ maps onto a difference codeword in $\mathcal{D}^*$, i.e.,

$$x \neq x' \Rightarrow C(x) \neq C(x').$$



Source Coding: Code types

Definition (Nonsingular Code)

A code C is called *nonsingular* if every element of $\mathcal{X}$ maps onto a difference string in $\mathcal{D}^*$, i.e.,

$$x \neq x' \Rightarrow C(x) \neq C(x').$$

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.

Source Coding: Code types

Definition (Code Extension)

The *extension* of a code C: $\mathcal{X} \to \mathcal{D}^*$ is defined by

$$C(x_1 x_2 \cdots x_n) = C(x_1)C(x_2)\cdots C(x_n).$$

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

$$x_1 x_2 \ldots x_m \neq x_1' x_2' \ldots x_n' \Rightarrow C\left(x_1 x_2 \ldots x_m\right) \neq C\left(x_1' x_2' \ldots x_n'\right)$$

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Source Coding: Code types

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

C_{II}^* is singular. $(C(1,1) = C(3) = 00)$

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

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Source Coding: Code types

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

C_{III} is uniquely decodable.

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

Source Coding: Code types

Definition (Unique Decodable Code)

A code is called *uniquely decodable* if its extension is nonsingular.

$$1100000000 = 3, 2, 2, 2, 2$$
$$11000000000 = 4, 2, 2, 2, 2$$

To know the source, we have to wait until the end!

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..

Source Coding: Code types

Definition (Prefix Code)

A code C is called a *prefix code* (a.k.a. *instantaneous*) iff no codeword of C is a prefix of any other codeword of C.

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

C_I is singular.
C_{II} is NOT u.d..
C_{III} is NOT prefix.
C_{IV} is prefix.

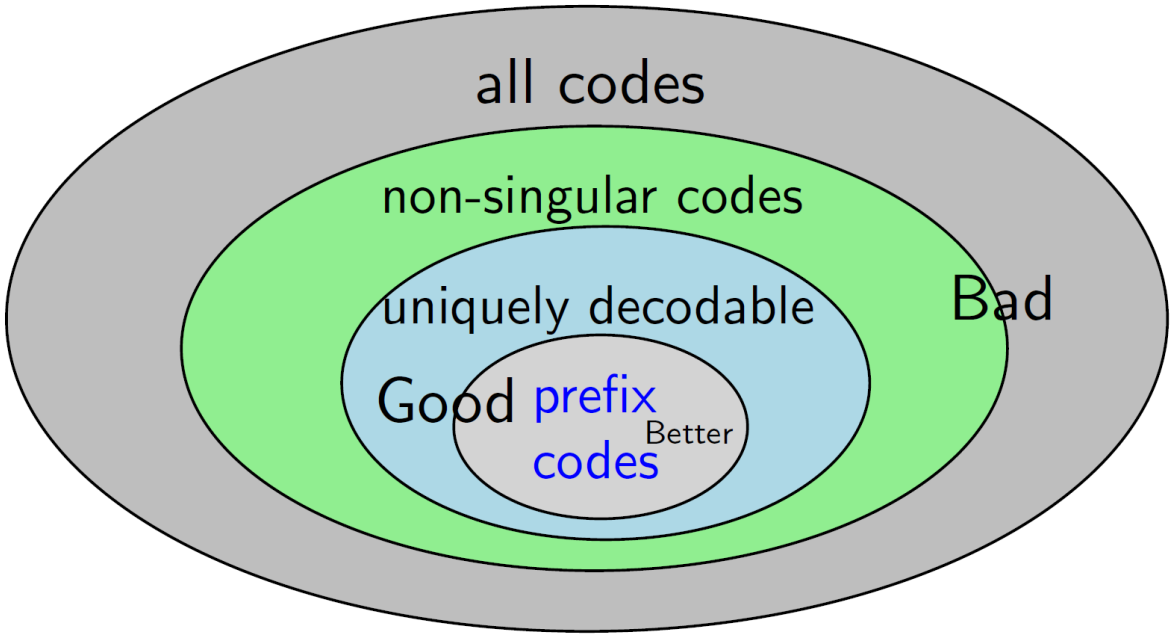
Source Coding: Code types

For $\mathcal{X} = \{1, 2, 3, 4\}$ and binary code, consider

x	$p(x)$	C_I	C_{II}	C_{III}	C_{IV}
1	1/2	0	0	10	0
2	1/4	0	1	00	10
3	1/8	1	00	11	110
4	1/8	10	11	110	111
$H(X)$	1.75	–	–	–	–
$E\ell(X)$	–	1.125	1.25	2.125	1.75

- C_I is singular.
- C_{II} is non-singular, but not uniquely decodable.
- C_{III} is non-singular, uniquely decodable, but NOT prefix.
- C_{IV} is non-singular, uniquely decodable, and prefix.

Source Coding: Classes of codes



- Goal: to find a prefix code with minimum expected length.

Kraft Inequality

Theorem 5.2.1 (Kraft Inequality)

For any prefix code over an alphabet of size D, the codeword lengths $\ell_1, \ell_2, \ldots, \ell_m$ must satisfy the inequality

$$\sum_i D^{-\ell_i} \leq 1.$$

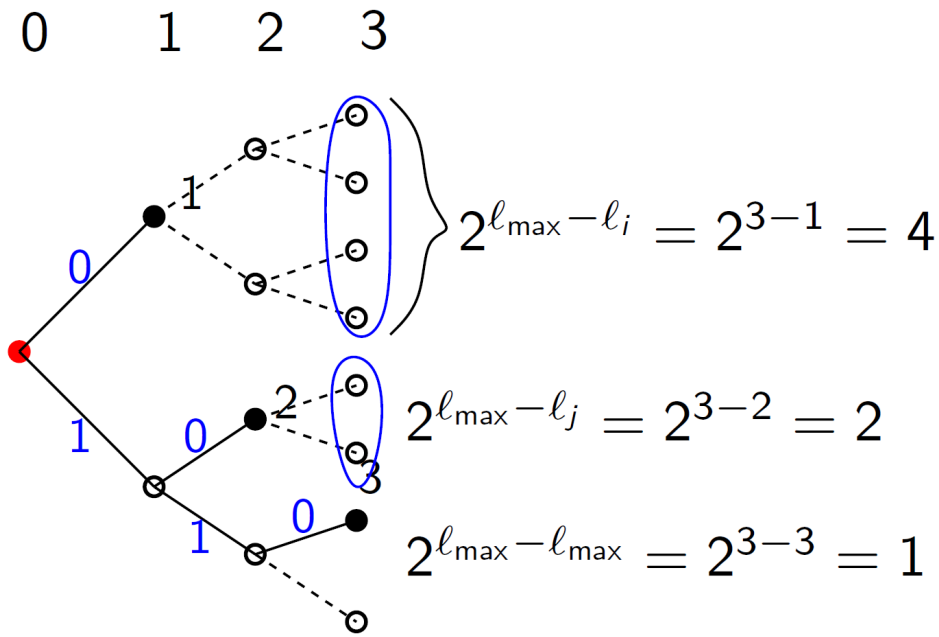
Conversely, given a set of codeword lengths that satisfy this inequality, there exists a prefix code with these codeword lengths.

Kraft Inequality

Proof Idea. (A small example) To prove: A prefix code with lengths $\ell_1, \ell_2, \ldots, \ell_m$, the inequality

$$\sum_i D^{-\ell_i} \leq 1 \qquad \text{holds.}$$

x	$c(x)$
1	0
2	10
3	110



$\ell_{\max} = 3$

$$\sum_i 2^{-\ell_i} \leq 1 \Leftarrow \sum_i 2^{\ell_{\max}-\ell_i} \leq 2^{\ell_{\max}}$$

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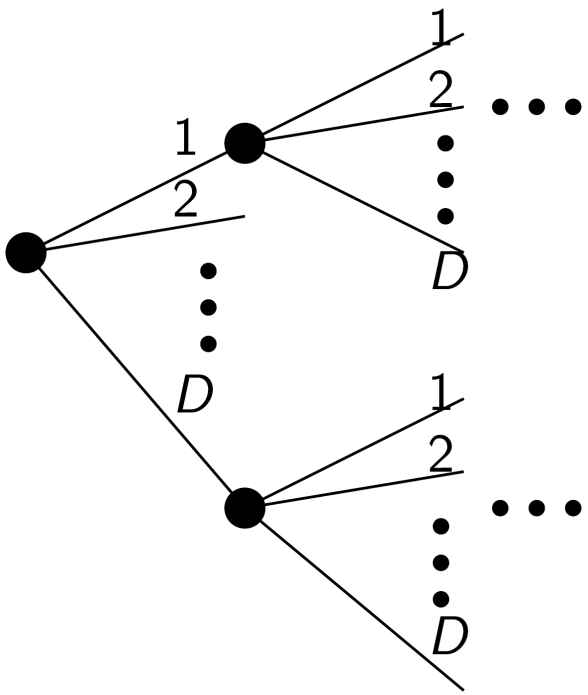
Kraft Inequality

Proof. (in general)

- Represent the set of prefix codes on a D-ary tree:

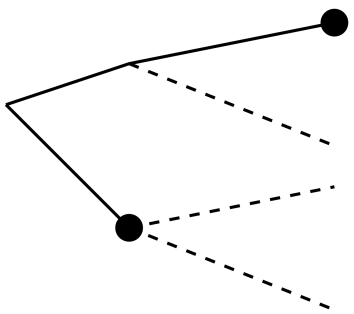


- Codewords correspond to leaves
- Path from root to each leaf determines a codeword
- Prefix condition: won't get to a codeword until we get to a leaf (no descendants of codewords are codewords)

Kraft Inequality

Proof. (in general)

- $\ell_{\max} = \max_i(\ell_i)$ is the length of the longest codeword.
- We can expand the full-tree down to depth $\ell_{\max}$:



The nodes at the level $\ell_{\max}$ are either

1. codewords
2. descendants of codewords
3. neither

- Consider a codeword i at depth ℓ_i in tree
- There are $D^{\ell_{\max}-\ell_i}$ descendants in the tree at depth $\ell_{\max}$
- Descendants of code i are disjoint from decedents of code j (prefix free condition)

Kraft Inequality

Proof. (in general)

- All the above implies:

$$\sum_i D^{\ell_{\max}-\ell_i} \leq D^{\ell_{\max}} \quad \Rightarrow \sum_i D^{-\ell_i} \leq 1$$

- Conversely: given codewords lengths $\ell_1, \ell_2, \ldots, \ell_m$ satisfying Kraft inequality, try to construct a prefix code.

Kraft Inequality

Proof. (in general)

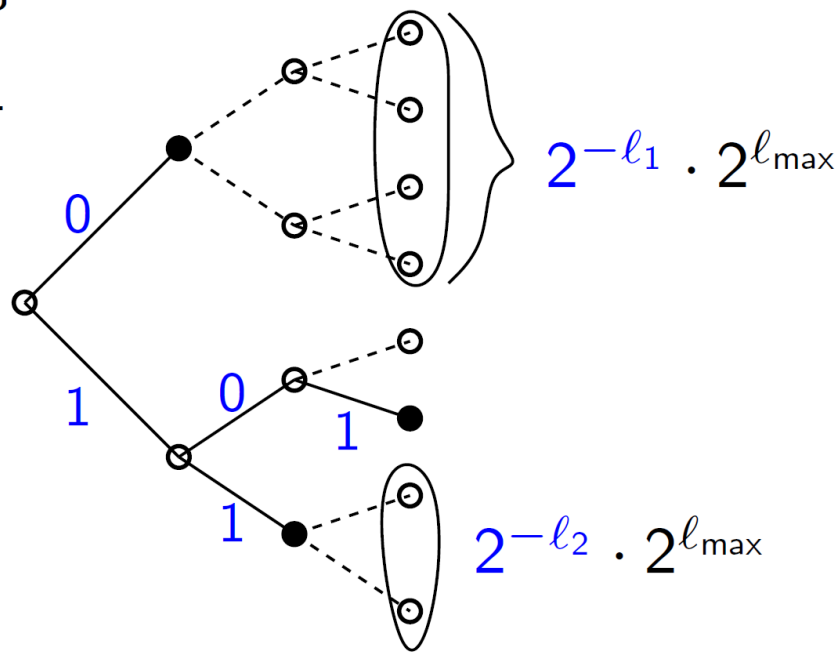
- Conversely: given codewords lengths $\ell_1, \ell_2, \ldots, \ell_m$ satisfying Kraft inequality, try to construct a prefix code.

$$\{\ell_1, \ell_2, \ell_3\} = \{1, 2, 3\}$$

$$2^{-1} + 2^{-2} + 2^{-3} \leq 1$$

x	$c(x)$
1	0
2	11
3	101

C is prefix.



Kraft Inequality

Proof. (in general)

- Conversely: given codewords lengths $\ell_1, \ell_2, \ldots, \ell_m$ satisfying Kraft inequality, try to construct a prefix code.

Left as an Exercise.

Extended Kraft Inequality

Theorem 5.2.2 (Extended Kraft Inequality)

Kraft inequality holds also for all countably infinite set of codewords, i.e., the codeword lengths satisfy the extended Kraft inequality,

$$\sum_{i=1}^{\infty} D^{-\ell_i} \leq 1$$

Conversely, given any $\ell_1, \ell_2, \ldots$ satisfying the extended Kraft inequality, we can construct a prefix code with these codeword lengths.

Extended Kraft Inequality

Theorem 5.2.2 (Extended Kraft Inequality)

Kraft inequality holds also for all countably infinite set of codewords.

Proof.

Consider the ith codeword $y_1 y_2 \cdots y_{\ell_i}$. Let $0.y_1 y_2 \cdots y_{\ell_i}$ be the real number given by the D-ary expansion

$$0.y_1 y_2 \cdots y_{\ell_i} = \sum_{j=1}^{\ell_i} y_j D^{-j},$$

which corresponds to the interval

$$[0.y_1 y_2 \cdots y_{\ell_i}, 0.y_1 y_2 \cdots y_{\ell_i} + \frac{1}{D^{\ell_i}}).$$

Extended Kraft Inequality

Theorem 5.2.2 (Extended Kraft Inequality)

Kraft inequality holds also for all countably infinite set of codewords.

Proof. (cont.)

By the prefix condition, these intervals are disjoint in the unit interval $[0, 1]$. Thus, the sum of their lengths is ≤ 1. This proves that

$$\sum_{i=1}^{\infty} D^{-\ell_i} \leq 1.$$

For converse, reorder indices in increasing order and assign intervals as we walk along the unit interval. □

Reading & Homework

Reading : 5.1, 5.2

Homework : Problems 5.1, 5.3